

The Cauchy problem for fully nonlinear parabolic systems on manifolds

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Abstract

We show the short time existence and uniqueness of solutions to the Cauchy problem for fully nonlinear systems of arbitrary even order on closed manifolds which are strongly parabolic at the initial values. The proof uses a linearization procedure and a fixed-point argument, and the key ingredient is the well known Schauder estimates for linear, strongly parabolic systems.

Key words: fully nonlinear parabolic systems, linearization, Schauder estimates, fixed-point

2010 Mathematics Subject Classification: 53C44, 35K46

1 Introduction

In this note we show the short time existence and uniqueness of solutions to the Cauchy problem for fully nonlinear systems of arbitrary even order on closed manifolds which are strongly parabolic at the initial values.

Let (M^n, g) be a closed (i.e. compact and without boundary) Riemannian manifold, and E be a smooth vector bundle of rank l over M with a metric h on the fibers and a connection ∇ compatible with h . Let $\Gamma(E)$ be the space of smooth sections of E , sometimes one also uses $C^\infty(E)$ to denote the same space. Let $J^r(E)$ be the r -th jet bundle of E , whose fiber over $x \in M$ is $J^r(E)_x = C^\infty(E)/Z_x^r(E)$, where $Z_x^r(E)$ is the ring of smooth sections u of E satisfying $(\nabla^k u)(x) = 0$ for all $0 \leq k \leq r$; see for example [P] and [S]. (Here we are slightly abusing notation: to define ∇^k ($k \geq 2$) one needs also the Levi-Civita connection of (M, g) ; cf. for example pp. 9-10 of [C].) Given $x \in M$, for any $u \in C^\infty(E)$, the corresponding class in $C^\infty(E)/Z_x^r(E)$ is denoted by $j_r(u)_x$. For $u \in C^r(E)$ and $x \in M$, we let $j_r(u)_x = j_r(\tilde{u})_x$ with $\tilde{u} \in C^\infty(E)$ satisfying $\nabla^k \tilde{u}(x) = \nabla^k u(x)$ for all $0 \leq k \leq r$.

Let $P_t : \text{Dom}(P_t) \rightarrow \Gamma(E)$, $t \in [0, T]$ be a (smooth family of) smooth partial differential operator(s) of order r , where $\text{Dom}(P_t)$ is a subset of $\Gamma(E)$. For $u \in \text{Dom}(P_t)$, we may write

$$P_t(u)(x) = F(x, t, u(x), \nabla u(x), \dots, \nabla^r u(x)) \in E_x, \quad (1.1)$$

where $x \in M$ and $t \in [0, T]$ for some $T > 0$. Here, for fixed t , F can be viewed as a map from a subset of $J^r(E)$ to E , see for example [P], [S] and [Hu]. In local

coordinates (x^1, \dots, x^n) of M and local frame $\{e_a\}_{a=1}^l$ of E ,

$$P_t(u)(x) = F^a(x^1, \dots, x^n, t, u^1(x), \dots, u^l(x), \frac{\partial u^1}{\partial x^1}(x), \dots, (\frac{\partial}{\partial x^n})^r u^l(x)) e_a(x), \quad (1.2)$$

where F^a are smooth (C^∞) functions of their arguments (for smooth P_t).

Given $u, v \in \Gamma(E)$, the linearization of the operator P_t at u in the direction v is $P_{t*|u}(v) = \frac{\partial}{\partial s}(P_t(u + sv))|_{s=0} = \lim_{s \rightarrow 0} \frac{P_t(u+sv) - P_t(u)}{s}$. Given a covector $(x, \xi) \in T^*M$, choose $\phi \in C^\infty(M)$ with $\phi(x) = 0$ and $d\phi(x) = \xi$. Given $v_x \in E_x$, choose $v \in C^\infty(E)$ with $v(x) = v_x$. Define the principal symbol $\sigma(P_{t*|u})$ of $P_{t*|u}$ via

$$\sigma(P_{t*|u})(x, \xi)v_x = \frac{1}{r!} P_{t*|u}(\phi^r v)(x).$$

Compare for example pp. 62-63 in [P] and pp. 45-46 in [To]. Let the order r of the operator P_t be even. The operator P_t is strongly elliptic at $u \in \Gamma(E)$ if there exists a constant $\lambda > 0$ such that

$$(-1)^{\frac{r}{2}-1} h(\sigma(P_{t*|u})(x, \xi)v_x, v_x) \geq \lambda |\xi|^r |v_x|^2$$

for all $(x, \xi) \in T^*M$ and $v_x \in E_x$. The above condition is often called the Legendre-Hadamard condition.

The above definitions can also be given with less smoothness assumption.

Let the vector bundle E be as above, u_0 be a given section of E , P_t be a (family of) differential operator(s) of (even) order r , consider the equation

$$\frac{\partial u}{\partial t} = P_t(u(\cdot, t)), \quad u(\cdot, 0) = u_0, \quad (1.3)$$

for sections $u(\cdot, t)$ of E .

The following result is ‘standard’.

Theorem 1.1. *Let P_t and u_0 be smooth. Suppose P_0 is strongly elliptic at u_0 . Then there is $\delta > 0$, such that the equation (1.3) has a unique smooth solution defined on $M \times [0, \delta]$.*

There are extensive researches on short time existence and uniqueness for solutions to quasilinear and fully nonlinear parabolic equations/systems in the literature, but most of them focus on equations/systems in domains in Euclidean spaces, see for example, the books [E1], [LSU], [F2], [Li], [Lu1] and the references therein; the corresponding works on manifolds are much less, we will only mention a few of them. For the quasilinear, second order case see, for example, Theorem 4.51 in Aubin [A] (cf. also Hamilton [H1]) and Section 8 in Chapter 15 of Taylor [T2], and for the quasilinear, higher order case see Mantegazza and Martinazzi [MM]. There is a result in Baker [B] (see Main Theorem 1 in [B], cf. also Theorem 2.4.5 in Lamm [L]) for the general fully nonlinear case similar to our Theorem 1.1, but Baker [B] imposes an extra symmetry condition on the linearized operator, and Lamm [L]

only considers trivial bundles. Also note that there is a result on fully nonlinear second order parabolic systems in Section 7.3 of [T1]. Theorem 1.1 is extended to transversely parabolic systems on foliated manifolds in Huang [Hu].

Many geometric evolution equations/systems are only degenerate parabolic, but by using the so called De Turck trick one can often convert them to strongly parabolic equations/systems, then one can apply Theorem 1.1 to get short time solutions. For example, Ricci flow, mean curvature flow, cross curvature flow and some fourth-order geometric flows are among them.

Theorem 1.1 follows from Theorem 1.2 below (with less smoothness in the assumption and conclusion) through a standard bootstrap argument. To state Theorem 1.2 we need to introduce some notations. For $0 < \alpha < 1$ let $C^{r+\alpha}(E)$ be the space of $C^{r+\alpha}$ -sections of the bundle E . For $0 < \tau \leq T$, slightly abusing notation, let $E_\tau \rightarrow M \times [0, \tau]$ be the pullback of the bundle $E \rightarrow M$ via the projection $M \times [0, \tau] \rightarrow M$, and $C^{r+\alpha, 1+\frac{\alpha}{\tau}}(E_\tau)$ be the space of $C^{r+\alpha, 1+\frac{\alpha}{\tau}}$ -sections of the bundle E_τ . Here to define the Hölder spaces for sections of vector bundles over manifolds we use parallel translation (defined via the connection ∇), see for example, [N] and [B] for the precise definition, which is an extension of that in Euclidean spaces (cf. [LSU], [Li], [Lu1] and [K]). (Note that instead of $C^{r+\alpha, 1+\frac{\alpha}{\tau}}$ here, Baker uses the notation $C^{r, 1, \alpha}$.)

Now let the (not necessarily C^∞ -smooth) differential operator P_t , $t \in [0, T]$, be given again by (1.1), where $F : B \times [0, T] \rightarrow E$ with

$$B = \{j_r(u)_x \mid x \in M, u \in C^r(E), \\ \|u - u_0\|_{C^0} + \|\nabla u - \nabla u_0\|_{C^0} + \cdots + \|\nabla^r u - \nabla^r u_0\|_{C^0} \leq R_0\} \subset J^r(E)$$

for some $u_0 \in C^{r+\alpha}(E)$ ($0 < \alpha < 1$) and $R_0 > 0$, and we suppose the map F is twice continuously differentiable (i.e., $F \in C^2(B \times [0, T], E)$), which means that F^a in (1.2) are C^2 -functions of their arguments.

Theorem 1.2. *Let P_t , F and u_0 be as in the preceding paragraph. Assume that P_0 is strongly elliptic at u_0 . Then there is $\delta > 0$, such that the equation (1.3) has a unique solution $u \in C^{r+\alpha, 1+\frac{\alpha}{\tau}}(E_\delta)$ defined on $M \times [0, \delta]$.*

Theorem 1.2 is somewhat sharper than the corresponding statements in Main Theorem 1 of Baker [B] and in Theorem 2.4.5 of Lamm [L] in that here we do not have any lose on the Hölder exponent in the conclusion. Moreover Theorem 2.4.5 of Lamm [L] has stronger assumption on the regularity of the map F . (Note that Lamm [L] considers the Cauchy-Dirichlet problem on compact manifolds with boundary, our results can also be extended to this case.)

Our proof of Theorem 1.2 mainly follows that of Theorem 8.5.4 in Lunardi's book [Lu1], which deals with fully nonlinear second order parabolic equations in domains in the Euclidean space. (Compare [AT], [B], [L], and [Lu2].) It uses a linearization procedure and a fixed-point argument, and the key ingredient is the well known Schauder estimates for linear, strongly parabolic systems due to Solonnikov [So] (compare Section 10 of Chapter VII in [LSU]). There are many

other works on the Schauder estimates for linear parabolic systems, see for example Eidelman [E1], Friedman [F1] [F2], Giaquinta-Modica [GM], Lamm [L], and Schlag [Sc]. For recent works on Schauder estimates on linear, strongly parabolic systems with less regularity in the coefficients, see for example, [Bo] and [DZ].

In Section 2 we briefly reformulate the linear parabolic Schauder theory on closed manifolds, in Section 3 we prove Theorem 1.2, and finally in Section 4 we prove Theorem 1.1.

2 The linear parabolic Schauder theory

Let the manifold M and the vector bundle E be as in the Introduction, and $L_t : C^{r+\alpha}(E) \rightarrow C^\alpha(E)$ ($t \in [0, T]$) be a family of linear, strongly elliptic operators of order r whose coefficients are of $C^{\alpha, \frac{\alpha}{r}}$ in any local chart. We fix a finite number of normal coordinate charts of (M, g) whose union is M , and fix a trivialization of E over each such chart. Then for a section u of E locally we may write $(L_t u)^a = \sum_{|I| \leq r, b \leq l} A_b^{aI}(x, t) \partial_I u^b$, where $\partial_I u^b = \frac{\partial^{|I|} u^b}{\partial x^{i_1} \dots \partial x^{i_j}}$ for $I = (i_1, \dots, i_j)$ (here $|I| = j$). (Our usage of the multi-index I is somewhat different from the usual convention, but it is convenient for us.) Let Λ be the maximum of $\sum_{|I| \leq r} \|A^I\|_{C^{\alpha, \frac{\alpha}{r}}}$ over the finitely many charts.

Let $u_0 \in C^{r+\alpha}(E)$ and $f \in C^{\alpha, \frac{\alpha}{r}}(E_T)$, where E_T is defined in the Introduction, consider the linear, strongly parabolic system

$$\frac{\partial u}{\partial t} = L_t(u(\cdot, t)) + f(\cdot, t), \quad u(\cdot, 0) = u_0, \quad (2.1)$$

on $M \times [0, T]$ for sections $u(\cdot, t)$ of E . In the case that E is a trivial bundle over a domain in \mathbb{R}^n we have the following crucial Schauder estimates (interior w.r.t. the space) for (2.1) due to Solonnikov [So].

Theorem 2.1. *Let Ω and Ω' be smooth bounded domains in \mathbb{R}^n with $\bar{\Omega}' \subset \Omega$, $Q = \Omega \times [0, T]$ and $Q' = \Omega' \times [0, T]$. Let $L_t : C^{r+\alpha}(\bar{\Omega}, \mathbb{R}^l) \rightarrow C^\alpha(\bar{\Omega}, \mathbb{R}^l)$ ($t \in [0, T]$) be a family of linear, strongly elliptic operator of order r with $(L_t u)^a = \sum_{|I| \leq r, b \leq l} A_b^{aI}(x, t) \partial_I u^b$, where $A_b^{aI}(\cdot, \cdot) \in C^{\alpha, \frac{\alpha}{r}}(\bar{Q})$. Fix $1 < p \leq \infty$. Then there is a constant C depending on Ω , λ , Λ , α , p and the distance between Ω' and $\partial\Omega$ such that given $u_0 \in C^{r+\alpha}(\bar{\Omega}, \mathbb{R}^l)$ and $f \in C^{\alpha, \frac{\alpha}{r}}(\bar{Q}, \mathbb{R}^l)$, if $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{Q}, \mathbb{R}^l)$ is a solution to the system (2.1) on Q with values in \mathbb{R}^l , then*

$$\|u\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(Q')} \leq C(\|f\|_{C^{\alpha, \frac{\alpha}{r}}(Q)} + \|u_0\|_{C^{r+\alpha}(\Omega)} + \|u\|_{L^p(Q)}). \quad (2.2)$$

Here, $\lambda > 0$ is the infimum of the constants in the Legendre-Hadamard condition for L_t over $t \in [0, T]$, and $\Lambda = \sum_{|I| \leq r} \|A^I\|_{C^{\alpha, \frac{\alpha}{r}}(Q)}$.

Proof. This is a special case of Theorem 4.11 in [So]. ([So] deals with systems parabolic in a more general sense.) In particular, note the last paragraph in the proof of Theorem 4.11 there for our case of ‘purely interior cylinder Q' ’. (Compare

the global Schauder estimates in Theorem 4.9 of [So], Theorem 3.1 in Section 1 of Chapter 3 in [E2] and Theorem VI.21 in [EZ].) \square

Remark If the coefficients of L_t , f and the initial value u_0 have higher regularity, then by Theorem 4.11 in [So] one has corresponding improved Schauder estimates. Also note that there is an estimate similar to (2.2) in Theorem 2.3.14 of Lamm [L], where the RHS of the estimate contains $\|u\|_{L^\infty}$ instead of $\|u\|_{L^p}$. See also Theorem 2.3.23 in [L] for the corresponding global Schauder estimate.

In the Introduction we mentioned that we can define the Hölder norms of sections of the bundle E over M via parallel translation. It turns out that we can alternatively define the Hölder norms of sections of E using their components in local trivializations for E as in the Euclidean case, as observed by many people, see for example Proposition 3.21 in [B], [Bu] and Chapter 2 in [Sz]. In effect, when one estimates the Hölder norm of a section of E one can compare directly the components of the section at two points in a local trivialization of E without using the parallel translation defined by the connection ∇ .

Lemma 2.2. *Let $k \geq 0$ be an integer and $0 < \alpha < 1$. The $C^{k,\alpha}$ -norms of sections of E defined using their components in local trivializations for E as in the Euclidean case are uniformly equivalent to the norms defined via parallel translation.*

Proof. The estimates on the supremum norm terms are easy, as locally we can express the covariant derivatives using the ordinary derivatives and the Christoffel symbols of the connection ∇ on E and of the Levi-Civita connection on (M, g) , and we can control these Christoffel symbols and their derivatives since we are on a compact manifold. (Compare for example Corollary 4.11 in [H2] and Theorem 1.3 in [He].) Now we treat the Hölder semi-norms. For simplicity we only write down the argument for the case $k = 0$, the general case being similar. Given $x_0 \in M$ and $V_0 \in E_{x_0}$, the parallel translation of V_0 along a geodesic γ emanating from x_0 is defined by solving the homogeneous linear first order ODE

$$\nabla_{\dot{\gamma}(s)} V(s) = 0, \quad V(0) = V_0 \quad (2.3)$$

for $V(s) \in E_{\gamma(s)}$. Since the connection ∇ is compatible with the metric h , we have

$$|V(s)| = |V_0| \quad (2.4)$$

for the solution $V(s)$ to the ODE (2.3). In the chosen local coordinates (x^1, \dots, x^n) for M and local frame $\{e_a\}$ for E , we can rewrite (2.3) as

$$\frac{dV^a(s)}{ds} + \Gamma_{ib}^a(\gamma(s)) \frac{d\gamma^i(s)}{ds} V^b(s) = 0, \quad V^a(0) = V_0^a, \quad (2.5)$$

where V^a ($1 \leq a \leq l$) (resp. γ^i ($1 \leq i \leq n$)) are the components of V (resp. γ) in the local frame $\{e_a\}$ (resp. local coordinates (x^1, \dots, x^n)), and $\nabla_{\frac{\partial}{\partial x^i}} e_a = \Gamma_{ia}^b e_b$.

From (2.4) and (2.5) we have

$$|V^a(s)| \leq C_1 |V_0|, \quad \text{and} \quad |V^a(s) - V_0^a| \leq C_2 |V_0| s,$$

where C_1 and C_2 are constants depending on (E, h, ∇) . Also note that we only use finitely many charts. Then the result follows. \square

Now we have the following Schauder estimates for linear, strongly parabolic systems on closed manifolds, building on the Euclidean case (Theorem 2.1).

Theorem 2.3. *There exists a constant C depending on M , E , λ , Λ , α and T , such that if $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(E_T)$ is a solution to the linear, strongly parabolic system (2.1) on $M \times [0, T]$, then*

$$\|u\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(E_T)} \leq C(\|f\|_{C^{\alpha, \frac{\alpha}{r}}(E_T)} + \|u_0\|_{C^{r+\alpha}(E)}). \quad (2.6)$$

Here, $\lambda > 0$ is the infimum of the constants in the Legendre-Hadamard condition for L_t over $t \in [0, T]$, and $\Lambda < \infty$ is the constant related to the $C^{\alpha, \frac{\alpha}{r}}$ -norms of the coefficients of L_t in the given normal coordinate charts and local trivializations as defined in the first paragraph of this section. Moreover, C increases w.r.t. T .

Proof First as usual we can reduce the proof of (2.6) to the special case $u_0 = 0$. So below we assume $u_0 = 0$. Let $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(E_T)$ be a solution to the linear, strongly parabolic system (2.1) with $u(\cdot, 0) = 0$. By Theorem 2.1 (with $p = 2$) and Lemma 2.2 we have

$$\|u\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(E_T)} \leq C(\|f\|_{C^{\alpha, \frac{\alpha}{r}}(E_T)} + \|u\|_{L^2(M \times [0, T])}), \quad (2.7)$$

where C depends on M , E , λ , Λ and α . (For the transition between the Euclidean case and the manifold case one can consult [Bu] and the proof of Proposition 3.22 in [B].) Below we'll use the same C to denote various constants different from line to line.

We try to get rid of the dependence on $\|u\|_{L^2(M \times [0, T])}$ on the RHS of (2.7) by allowing the constant C to depend also on T . (Compare the sentence that follows the inequality (66) on p. 1169 in [Sc].) For $s \in [0, T]$, let

$$v(s) = \int_M |u(x, s)|^2 d\mu.$$

Using $u(x, s) = \int_0^s \frac{\partial u}{\partial t} dt$ and (2.7) we have

$$v(s) \leq s^2 \text{vol}(M) \left\| \frac{\partial u}{\partial t} \right\|_{C^0(E_s)}^2 \leq C(\|f\|_{C^{\alpha, \frac{\alpha}{r}}(E_s)}^2 + \int_0^s v(t) dt),$$

where the constant C depends also on s , and increases w.r.t. s . By Bellman-Gronwall inequality we get

$$v(s) \leq C \|f\|_{C^{\alpha, \frac{\alpha}{r}}(E_s)}^2.$$

It follows that

$$\|u\|_{L^2(M \times [0, T])} \leq C \|f\|_{C^{\alpha, \frac{\alpha}{r}}(E_T)},$$

which, combined with (2.7), implies the desired estimate. \square

Remark There is a similar estimate in Lemma 2.3.25 of Lamm [L] in the case that E is a trivial bundle. But Lamm's proof there uses the parabolic L^2 -theory (Theorem 2.2.1 in [L]) which needs stronger assumption on the regularity of the coefficients of the operator L_t to be able to write L_t in divergence form.

Then, based on Theorem 2.3, we have

Theorem 2.4. *Let the coefficients of the linear, strongly elliptic operator L_t ($t \in [0, T]$) be of $C^{\alpha, \frac{\alpha}{r}}$ in any local chart, $f \in C^{\alpha, \frac{\alpha}{r}}(E_T)$ and $u_0 \in C^{r+\alpha}(E)$. Then there exists a unique solution $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(E_T)$ to the linear, strongly parabolic system (2.1) on $M \times [0, T]$.*

Proof. The proof is by a standard procedure, compare the proof of Proposition 3.25 in [B] and Theorem 2.4.3 in [L]. First one solves the simpler system

$$\frac{\partial u}{\partial t} = (-1)^{\frac{r}{2}-1} \Delta^{r/2}(u(\cdot, t)) + f(\cdot, t), \quad u(\cdot, 0) = u_0$$

via L^2 -theory, where $\Delta = \text{tr}_g \nabla^2 = -\nabla^* \nabla$ is the connection (rough) Laplacian defined by using ∇ and the Levi-Civita connection of (M, g) (see for example, pp. 9-11 in [C]). (Note that the results in Sections 2.2 and 2.3 of Chapter 2 in [Po] can be easily extended to our situation. In particular, Garding's inequality (cf. Lemma 2.2.1 in [Po]) also holds for our operator $(-1)^{\frac{r}{2}} \Delta^{r/2}$ due to its special structure, that is, there is a constant C depending on (M, g) and (E, h, ∇) such that for any $\psi \in W^{\frac{r}{2}, 2}(E)$, there holds

$$\int_M h((-1)^{\frac{r}{2}} \Delta^{r/2} \psi, \psi) d\mu \geq \frac{1}{2} \|\psi\|_{W^{\frac{r}{2}, 2}(E)}^2 - C \|\psi\|_{L^2(E)}^2.$$

Here we need the compatibility of the connection ∇ with the fiber metric h .) Then using the method of continuity and the Schauder estimates as in Theorem 2.3 one solves the system (2.1).

The uniqueness follows from the Schauder estimate (Theorem 2.3) for the homogeneous, linear, strongly parabolic system with initial value zero, which is satisfied by the difference of any two solutions in $C^{r+\alpha, 1+\frac{\alpha}{r}}(E_T)$ to the system (2.1). \square

When the given data have higher regularity we have the following

Theorem 2.5. *Let $k \geq 0$ be an integer. Let the coefficients of the linear, strongly elliptic operator L_t be of $C^{k+\alpha, \frac{k+\alpha}{r}}$ in any local chart, $f \in C^{k+\alpha, \frac{k+\alpha}{r}}(E_T)$ and $u_0 \in C^{r+k+\alpha}(E)$. Then there exists a unique solution $u \in C^{r+k+\alpha, 1+\frac{k+\alpha}{r}}(E_T)$ to the linear, strongly parabolic system (2.1) on $M \times [0, T]$, and*

$$\|u\|_{C^{r+k+\alpha, 1+\frac{k+\alpha}{r}}(E_T)} \leq C(\|f\|_{C^{k+\alpha, \frac{k+\alpha}{r}}(E_T)} + \|u_0\|_{C^{r+k+\alpha}(E)}), \quad (2.8)$$

where the constant C depends on $M, E, \lambda, \Lambda, \alpha$ and T , and C increases w.r.t. T . Here, $\lambda > 0$ is the infimum of the constants in the Legendre-Hadamard condition for L_t over $t \in [0, T]$, and $\Lambda < \infty$, a constant related to the $C^{k+\alpha, \frac{k+\alpha}{r}}$ -norms of the coefficients of L_t in the given normal coordinate charts and local trivializations, is defined in a way similar to that in the first paragraph of this section.

Proof. The proof is similar to that of Theorems 2.3 and 2.4, using Theorem 4.11 of [So]. \square

Consequently we have the following

Theorem 2.6. *Let the linear, strongly elliptic operator L_t ($t \in [0, T]$), f and u_0 be smooth (C^∞). Then there exists a unique smooth solution to the linear, strongly parabolic system (2.1) on $M \times [0, T]$.*

3 Proof of Theorem 1.2

We'll need the following elementary interpolation inequality which extends Lemma 7 in Schlag [Sc]; compare also Lemma 2.3.4 in [L].

Lemma 3.1. *Let Ω be a $C^{r+\alpha}$ domain in \mathbb{R}^n , and $Q = \Omega \times [0, T]$. Then for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, r, \Omega)$ not depending on T , such that for any $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{Q}, \mathbb{R}^l)$,*

$$\begin{aligned} & |\partial_x^r u|_{0;Q} + |\partial_x^{r-1} u|_{0;Q} + [\partial_x^{r-1} u]_{\alpha;Q} + \cdots + |\partial_x u|_{0;Q} + [\partial_x u]_{\alpha;Q} + [u]_{\alpha;Q} \\ & \leq \varepsilon [\partial_x^r u]_{\alpha;Q} + \varepsilon [\partial_t u]_{\alpha;Q} + C |u|_{0;Q}. \end{aligned}$$

Here, $\partial_x^i u$ means any i -th order derivative of u w.r.t. the space variables.

Proof. The proof is similar to that of Lemma 7 in [Sc], compare also the proof of Lemma 6.35 in [GT] and Lemma 2.3.4 in [L]. \square

Now we prove Theorem 1.2. Given $u_0 \in C^{r+\alpha}(E)$ as in the assumption of Theorem 1.2. Let $\min\{1, T\} \geq \delta > 0$ and $R > 0$ be constants to be chosen later (to satisfy three conditions below), and

$$Y = \{u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(E_\delta) \mid u(\cdot, 0) = u_0, \|u - u_0\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} \leq R\}.$$

Using Lemma 3.1 and a standard procedure via local trivializations for the bundle E (cf. Lemma 2.2 and its proof), we see that there is a positive constant C depending on M, E and r but not on δ or R , such that for any $u \in Y$,

$$\begin{aligned} & \|u - u_0\|_{C^0} + \|\nabla u - \nabla u_0\|_{C^0} + \cdots + \|\nabla^r u - \nabla^r u_0\|_{C^0} \\ & \leq C \delta^{\frac{\alpha}{r}} \|u - u_0\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(E_\delta)} \leq C \delta^{\frac{\alpha}{r}} R. \end{aligned}$$

The first condition that we want to impose on δ and R is

$$C \delta^{\frac{\alpha}{r}} R \leq \frac{R_0}{2}. \tag{3.1}$$

Then it makes sense to write $F(\cdot, t, u(\cdot, t), \dots, \nabla^r u(\cdot, t))$ for $u \in Y$ and $t \in [0, \delta]$.

As in the proof of Theorem 8.5.4 in [Lu1], we go to define a map $G : Y \rightarrow Y$ by setting $G(u) = w$ for $u \in Y$, where w is the unique solution in $C^{r+\alpha, 1+\frac{\alpha}{r}}(E_\delta)$ to the linear, strongly parabolic system

$$\begin{aligned} \frac{\partial w}{\partial t} &= P_{0*|u_0}(w(\cdot, t)) + F(\cdot, t, u(\cdot, t), \nabla u(\cdot, t), \dots, \nabla^r u(\cdot, t)) - P_{0*|u_0}(u(\cdot, t)), \\ w(\cdot, 0) &= u_0 \end{aligned}$$

on $M \times [0, \delta]$ given by Theorem 2.4.

We'll show that for $u, v \in Y$,

$$\|G(u) - G(v)\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} \leq C(R)\delta^{\frac{\alpha}{r}}\|u - v\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}}. \quad (3.2)$$

The second condition for δ and R is

$$C(R)\delta^{\frac{\alpha}{r}} \leq \frac{1}{2}. \quad (3.3)$$

Then, using the triangle inequality, (3.2) and (3.3), we have

$$\|G(u) - u_0\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} \leq \frac{R}{2} + \|G(u_0) - u_0\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}}. \quad (3.4)$$

But $\hat{v} := G(u_0) - u_0$ satisfies

$$\begin{aligned} \frac{\partial \hat{v}}{\partial t} &= P_{0*|u_0}(\hat{v}(\cdot, t)) + F(\cdot, t, u_0, \nabla u_0, \dots, \nabla^r u_0), \\ \hat{v}(\cdot, 0) &= 0. \end{aligned}$$

By Theorem 2.3, there is a constant C independent of δ such that

$$\|\hat{v}\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} \leq C\|F(\cdot, \cdot, u_0, \nabla u_0, \dots, \nabla^r u_0)\|_{C^{\alpha, \frac{\alpha}{r}}} =: C'. \quad (3.5)$$

(3.4) and (3.5) imply that

$$\|G(u) - u_0\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} \leq \frac{R}{2} + C'. \quad (3.6)$$

The third condition for R (and δ) is that R is suitably large with

$$\frac{R}{2} \geq C'. \quad (3.7)$$

Now suppose that R and δ satisfy all the three conditions (3.1), (3.3) and (3.7), and suppose that (3.2) is true. Then $G : Y \rightarrow Y$ is well-defined and is a $\frac{1}{2}$ -contraction. So by the contraction mapping principle G has a unique fixed-point, which clearly solves (1.3).

It remains to show (3.2).

Note that for $u, v \in Y$, $\tilde{w} := G(u) - G(v)$ satisfies

$$\begin{aligned}\frac{\partial \tilde{w}}{\partial t} &= P_{0*|u_0}(\tilde{w}(\cdot, t)) + F(\cdot, t, u(\cdot, t), \nabla u(\cdot, t), \dots, \nabla^r u(\cdot, t)) \\ &\quad - F(\cdot, t, v(\cdot, t), \nabla v(\cdot, t), \dots, \nabla^r v(\cdot, t)) - P_{0*|u_0}(u(\cdot, t) - v(\cdot, t)), \\ \tilde{w}(\cdot, 0) &= 0.\end{aligned}$$

By Theorem 2.3 again, there exists a constant C independent of δ such that

$$\|\tilde{w}\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} \leq C\|\eta\|_{C^{\alpha, \frac{\alpha}{r}}}, \quad (3.8)$$

where

$$\begin{aligned}\eta(x, t) &= F(x, t, u(x, t), \nabla u(x, t), \dots, \nabla^r u(x, t)) \\ &\quad - F(x, t, v(x, t), \nabla v(x, t), \dots, \nabla^r v(x, t)) - P_{0*|u_0}(u(x, t) - v(x, t)).\end{aligned}$$

We'll show

$$\|\eta\|_{C^{\alpha, \frac{\alpha}{r}}} \leq C(R)\delta^{\frac{\alpha}{r}}\|u - v\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}}. \quad (3.9)$$

Combined with (3.8), it implies (3.2).

By Lemma 2.2, we can estimate the Hölder norms of η and $u - v$ using their components in each local trivialization for E as in the Euclidean case. So below we assume that $E = \bar{\Omega} \times \mathbb{R}^l$, where $\bar{\Omega}$ is a smooth, compact domain (say a ball) in \mathbb{R}^n , and F is a C^2 -map from $\bar{\Omega} \times [0, \delta] \times Z$ to \mathbb{R}^l , where Z is a smooth, compact neighborhood of the range of $(u_0, \nabla u_0, \dots, \nabla^r u_0)$ in $\mathbb{R}^l \times \mathbb{R}^{nl} \times \dots \times \mathbb{R}^{n^r l}$.

For any $v \in C^r(E)$, we compute the linearization of P_0 at u_0 in the direction v ,

$$\begin{aligned}P_{0*|u_0}(v) &= F'_{i_1 \dots i_r}(x, 0, u_0, \nabla u_0, \dots, \nabla^r u_0) \nabla_{i_1 \dots i_r}^r v + \dots \\ &\quad + F'_k(x, 0, u_0, \nabla u_0, \dots, \nabla^r u_0) \nabla_k v + F'_u(x, 0, u_0, \nabla u_0, \dots, \nabla^r u_0) v.\end{aligned}$$

Here, by $F'_{i_1 \dots i_r}$ we mean the first order derivative of F w.r.t. the 'intermediate variable' $\nabla_{i_1 \dots i_r}^r u$, etc.

For $|\beta| = 1$, let $D^\beta F$ be any first order derivative of F w.r.t. $z \in Z$. (We'll interpret $D^0 F$ as F itself.)

Since F is C^2 , there exists a constant K_1 independent of δ such that

$$\sup\{\|D^\beta F(\cdot, \cdot, z)\|_{C^{\alpha, \frac{\alpha}{r}}} \mid z \in Z, |\beta| = 0, 1\} \leq K_1.$$

Similarly there also exists a constant K_2 independent of δ such that

$$\begin{aligned}|D^\beta F(x, t, z_1) - D^\beta F(x, t, z_2)| &\leq K_2|z_1 - z_2|, \\ \forall (x, t) \in \bar{\Omega} \times [0, \delta], \quad z_1, z_2 \in Z, \quad |\beta| = 0, 1.\end{aligned}$$

Then the remaining estimates are very similar to those in the proof of Theorem 8.5.4 in [Lu1]. But for completeness we'll reproduce them below.

For $\sigma \in [0, 1]$, let

$$\begin{aligned}\xi_\sigma(x, t) &= \sigma(u(x, t), \nabla u(x, t), \dots, \nabla^r u(x, t)) \\ &\quad + (1 - \sigma)(v(x, t), \nabla v(x, t), \dots, \nabla^r v(x, t)), \\ \xi_0(x) &= (u_0(x), \nabla u_0(x), \dots, \nabla^r u_0(x)).\end{aligned}$$

Note that there is a constant $C > 0$ such that

$$\begin{aligned} |\xi_\sigma(x, t) - \xi_\sigma(y, t)| &\leq C(\|u_0\|_{C^{r+\alpha}(E)} + R)|x - y|^\alpha, \\ |\xi_0(x) - \xi_0(y)| &\leq C\|u_0\|_{C^{r+\alpha}(E)}|x - y|^\alpha. \end{aligned}$$

We have

$$\begin{aligned} &F(x, t, u(x, t), \nabla u(x, t), \dots, \nabla^r u(x, t)) - F(x, t, v(x, t), \nabla v(x, t), \dots, \nabla^r v(x, t)) \\ &= \int_0^1 \frac{d}{d\sigma} F(x, t, \xi_\sigma(x, t)) d\sigma \\ &= \int_0^1 F'_{i_1 \dots i_r}(x, t, \xi_\sigma(x, t)) (\nabla_{i_1 \dots i_r}^r u(x, t) - \nabla_{i_1 \dots i_r}^r v(x, t)) d\sigma + \dots \\ &+ \int_0^1 F'_k(x, t, \xi_\sigma(x, t)) (\nabla_k u(x, t) - \nabla_k v(x, t)) d\sigma \\ &+ \int_0^1 F'_u(x, t, \xi_\sigma(x, t)) (u(x, t) - v(x, t)) d\sigma, \end{aligned}$$

and

$$\begin{aligned} \eta(x, t) &= \int_0^1 (F'_{i_1 \dots i_r}(x, t, \xi_\sigma(x, t)) - F'_{i_1 \dots i_r}(x, 0, \xi_0(x))) \nabla_{i_1 \dots i_r}^r (u - v)(x, t) d\sigma \\ &+ \dots + \int_0^1 (F'_k(x, t, \xi_\sigma(x, t)) - F'_k(x, 0, \xi_0(x))) \nabla_k (u - v)(x, t) d\sigma \\ &+ \int_0^1 (F'_u(x, t, \xi_\sigma(x, t)) - F'_u(x, 0, \xi_0(x))) (u - v)(x, t) d\sigma. \end{aligned}$$

In order to estimate $|\eta(x, t) - \eta(x, s)|$ for $0 \leq s \leq t \leq \delta$, we add and subtract

$$\begin{aligned} &\int_0^1 F'_{i_1 \dots i_r}(x, s, \xi_\sigma(x, s)) \nabla_{i_1 \dots i_r}^r (u - v)(x, t) d\sigma \\ &+ \dots + \int_0^1 F'_k(x, s, \xi_\sigma(x, s)) \nabla_k (u - v)(x, t) d\sigma \\ &+ \int_0^1 F'_u(x, s, \xi_\sigma(x, s)) (u - v)(x, t) d\sigma, \end{aligned}$$

and write

$$\begin{aligned} &\eta(x, t) - \eta(x, s) \\ &= \int_0^1 (F'_{i_1 \dots i_r}(x, t, \xi_\sigma(x, t)) - F'_{i_1 \dots i_r}(x, s, \xi_\sigma(x, s))) \nabla_{i_1 \dots i_r}^r (u - v)(x, t) d\sigma \\ &+ \dots + \int_0^1 (F'_k(x, t, \xi_\sigma(x, t)) - F'_k(x, s, \xi_\sigma(x, s))) \nabla_k (u - v)(x, t) d\sigma \\ &+ \int_0^1 (F'_u(x, t, \xi_\sigma(x, t)) - F'_u(x, s, \xi_\sigma(x, s))) (u - v)(x, t) d\sigma \\ &+ \int_0^1 (F'_{i_1 \dots i_r}(x, s, \xi_\sigma(x, s)) - F'_{i_1 \dots i_r}(x, 0, \xi_0(x))) \\ &\cdot (\nabla_{i_1 \dots i_r}^r (u - v)(x, t) - \nabla_{i_1 \dots i_r}^r (u - v)(x, s)) d\sigma \\ &+ \dots + \int_0^1 (F'_k(x, s, \xi_\sigma(x, s)) - F'_k(x, 0, \xi_0(x))) \\ &\cdot (\nabla_k (u - v)(x, t) - \nabla_k (u - v)(x, s)) d\sigma \\ &+ \int_0^1 (F'_u(x, s, \xi_\sigma(x, s)) - F'_u(x, 0, \xi_0(x))) ((u - v)(x, t) - (u - v)(x, s)) d\sigma. \end{aligned}$$

Thus we need to estimate

$$|D^\beta F(x, t, \xi_\sigma(x, t)) - D^\beta F(x, s, \xi_\sigma(x, s))|$$

and

$$|D^\beta F(x, s, \xi_\sigma(x, s)) - D^\beta F(x, 0, \xi_0(x))|$$

for $|\beta| = 1$.

We have

$$\begin{aligned}
& |D^\beta F(x, t, \xi_\sigma(x, t)) - D^\beta F(x, s, \xi_\sigma(x, s))| \\
& \leq |D^\beta F(x, t, \xi_\sigma(x, t)) - D^\beta F(x, s, \xi_\sigma(x, t))| \\
& + |D^\beta F(x, s, \xi_\sigma(x, t)) - D^\beta F(x, s, \xi_\sigma(x, s))| \\
& \leq (K_1 + K_2([\nabla^r u]_{C^{0, \frac{\alpha}{r}}} + [\nabla^r v]_{C^{0, \frac{\alpha}{r}}} \\
& + \cdots + [\nabla u]_{C^{0, \frac{\alpha}{r}}} + [\nabla v]_{C^{0, \frac{\alpha}{r}}} + [u]_{C^{0, \frac{\alpha}{r}}} + [v]_{C^{0, \frac{\alpha}{r}}})(t - s)^{\frac{\alpha}{r}} \\
& \leq C_1(R)(t - s)^{\frac{\alpha}{r}},
\end{aligned}$$

and similarly

$$\begin{aligned}
& |D^\beta F(x, s, \xi_\sigma(x, s)) - D^\beta F(x, 0, \xi_0(x))| \\
& \leq K_1 s^{\frac{\alpha}{r}} + K_2([\nabla^r(u - u_0)]_{C^{0, \frac{\alpha}{r}}} + [\nabla^r(v - u_0)]_{C^{0, \frac{\alpha}{r}}} + \cdots \\
& + [\nabla(u - u_0)]_{C^{0, \frac{\alpha}{r}}} + [\nabla(v - u_0)]_{C^{0, \frac{\alpha}{r}}} + [u - u_0]_{C^{0, \frac{\alpha}{r}}} + [v - u_0]_{C^{0, \frac{\alpha}{r}}})\delta^{\frac{\alpha}{r}} \\
& \leq C_2(R)\delta^{\frac{\alpha}{r}}.
\end{aligned}$$

Now

$$\begin{aligned}
& |\eta(x, t) - \eta(x, s)| \\
& \leq C_1(R)(t - s)^{\frac{\alpha}{r}}(\delta^{\frac{\alpha}{r}} \sum_{i_1, \dots, i_r=1}^n [\nabla_{i_1 \dots i_r}^r(u - v)]_{C^{0, \frac{\alpha}{r}}} + \cdots \\
& + \delta^{\frac{\alpha}{r}} \sum_{k=1}^n [\nabla_k(u - v)]_{C^{0, \frac{\alpha}{r}}} + \delta^{\frac{\alpha}{r}} [u - v]_{C^{0, \frac{\alpha}{r}}}) \\
& + C_2(R)\delta^{\frac{\alpha}{r}}((t - s)^{\frac{\alpha}{r}} \sum_{i_1, \dots, i_r=1}^n [\nabla_{i_1 \dots i_r}^r(u - v)]_{C^{0, \frac{\alpha}{r}}} + \cdots \\
& + (t - s)^{\frac{\alpha}{r}} \sum_{k=1}^n [\nabla_k(u - v)]_{C^{0, \frac{\alpha}{r}}} + (t - s)^{\frac{\alpha}{r}} [u - v]_{C^{0, \frac{\alpha}{r}}}) \\
& \leq C_3(R)\delta^{\frac{\alpha}{r}}(t - s)^{\frac{\alpha}{r}} \|u - v\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}}.
\end{aligned}$$

Letting $s = 0$, it implies that

$$\|\eta\|_{C^0} \leq C_3(R)\delta^{\frac{2\alpha}{r}} \|u - v\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}}.$$

In order to estimate $|\eta(x, t) - \eta(y, t)|$, we add and subtract

$$\begin{aligned}
& \int_0^1 (F'_{i_1 \dots i_r}(y, t, \xi_\sigma(y, t)) - F'_{i_1 \dots i_r}(y, 0, \xi_0(y))) \nabla_{i_1 \dots i_r}^r(u - v)(x, t) d\sigma \\
& + \cdots + \int_0^1 (F'_k(y, t, \xi_\sigma(y, t)) - F'_k(y, 0, \xi_0(y))) \nabla_k(u - v)(x, t) d\sigma \\
& + \int_0^1 (F'_u(y, t, \xi_\sigma(y, t)) - F'_u(y, 0, \xi_0(y))) (u - v)(x, t) d\sigma,
\end{aligned}$$

and write

$$\begin{aligned}
& \eta(x, t) - \eta(y, t) \\
& = \int_0^1 (F'_{i_1 \dots i_r}(x, t, \xi_\sigma(x, t)) - F'_{i_1 \dots i_r}(y, t, \xi_\sigma(y, t))) \nabla_{i_1 \dots i_r}^r(u - v)(x, t) d\sigma \\
& + \cdots + \int_0^1 (F'_k(x, t, \xi_\sigma(x, t)) - F'_k(y, t, \xi_\sigma(y, t))) \nabla_k(u - v)(x, t) d\sigma \\
& + \int_0^1 (F'_u(x, t, \xi_\sigma(x, t)) - F'_u(y, t, \xi_\sigma(y, t))) (u - v)(x, t) d\sigma \\
& + \int_0^1 (F'_{i_1 \dots i_r}(y, t, \xi_\sigma(y, t)) - F'_{i_1 \dots i_r}(y, 0, \xi_0(y))) \\
& \cdot (\nabla_{i_1 \dots i_r}^r(u - v)(x, t) - \nabla_{i_1 \dots i_r}^r(u - v)(y, t)) d\sigma + \cdots \\
& + \int_0^1 (F'_k(y, t, \xi_\sigma(y, t)) - F'_k(y, 0, \xi_0(y))) (\nabla_k(u - v)(x, t) - \nabla_k(u - v)(y, t)) d\sigma \\
& + \int_0^1 (F'_u(y, t, \xi_\sigma(y, t)) - F'_u(y, 0, \xi_0(y))) ((u - v)(x, t) - (u - v)(y, t)) d\sigma \\
& + \int_0^1 (F'_{i_1 \dots i_r}(y, 0, \xi_0(y)) - F'_{i_1 \dots i_r}(x, 0, \xi_0(x))) \nabla_{i_1 \dots i_r}^r(u - v)(x, t) d\sigma \\
& + \cdots + \int_0^1 (F'_k(y, 0, \xi_0(y)) - F'_k(x, 0, \xi_0(x))) \nabla_k(u - v)(x, t) d\sigma \\
& + \int_0^1 (F'_u(y, 0, \xi_0(y)) - F'_u(x, 0, \xi_0(x))) (u - v)(x, t) d\sigma.
\end{aligned}$$

Then we estimate

$$\begin{aligned}
& |D^\beta F(x, t, \xi_\sigma(x, t)) - D^\beta F(y, t, \xi_\sigma(y, t))| + |D^\beta F(x, 0, \xi_0(x)) - D^\beta F(y, 0, \xi_0(y))| \\
& \leq |D^\beta F(x, t, \xi_\sigma(x, t)) - D^\beta F(y, t, \xi_\sigma(x, t))| \\
& + |D^\beta F(y, t, \xi_\sigma(x, t)) - D^\beta F(y, t, \xi_\sigma(y, t))| \\
& + |D^\beta F(x, 0, \xi_0(x)) - D^\beta F(y, 0, \xi_0(x))| + |D^\beta F(y, 0, \xi_0(x)) - D^\beta F(y, 0, \xi_0(y))| \\
& \leq (K_1 + K_2([\nabla^r u]_{C^{\alpha,0}} + [\nabla^r v]_{C^{\alpha,0}} + \cdots + [\nabla u]_{C^{\alpha,0}} + [\nabla v]_{C^{\alpha,0}} + [u]_{C^{\alpha,0}} + [v]_{C^{\alpha,0}})) \\
& + K_1 + K_2([\nabla^r u_0]_{C^{\alpha,0}} + \cdots + [\nabla u_0]_{C^{\alpha,0}} + [u_0]_{C^{\alpha,0}})|x - y|^\alpha \\
& \leq C_4(R)|x - y|^\alpha,
\end{aligned}$$

and

$$\begin{aligned}
& |D^\beta F(y, t, \xi_\sigma(y, t)) - D^\beta F(y, 0, \xi_0(y))| \\
& \leq K_1 \delta^{\frac{\alpha}{r}} + K_2([\nabla^r(u - u_0)]_{C^{0,\frac{\alpha}{r}}} + [\nabla^r(v - v_0)]_{C^{0,\frac{\alpha}{r}}} + \cdots \\
& + [\nabla(u - u_0)]_{C^{0,\frac{\alpha}{r}}} + [\nabla(v - v_0)]_{C^{0,\frac{\alpha}{r}}} + [u - u_0]_{C^{0,\frac{\alpha}{r}}} + [v - v_0]_{C^{0,\frac{\alpha}{r}}})\delta^{\frac{\alpha}{r}} \\
& \leq C_5(R)\delta^{\frac{\alpha}{r}}.
\end{aligned}$$

Now

$$\begin{aligned}
& |\eta(x, t) - \eta(y, t)| \\
& \leq C_4(R)(\delta^{\frac{\alpha}{r}} \sum_{i_1, \dots, i_r=1}^n [\nabla_{i_1 \dots i_r}^r(u - v)]_{C^{0,\frac{\alpha}{r}}} + \cdots \\
& + \delta^{\frac{\alpha}{r}} \sum_{k=1}^n [\nabla_k(u - v)]_{C^{0,\frac{\alpha}{r}}} + \delta^{\frac{\alpha}{r}} [u - v]_{C^{0,\frac{\alpha}{r}}})|x - y|^\alpha \\
& + C_5(R)\delta^{\frac{\alpha}{r}} (\sum_{i_1, \dots, i_r=1}^n [\nabla_{i_1 \dots i_r}^r(u - v)]_{C^{\alpha,0}} + \cdots \\
& + \sum_{k=1}^n [\nabla_k(u - v)]_{C^{\alpha,0}} + [u - v]_{C^{\alpha,0}})|x - y|^\alpha \\
& \leq C_6(R)\delta^{\frac{\alpha}{r}} \|u - v\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}} |x - y|^\alpha.
\end{aligned}$$

Finally combining the above estimates for η we get

$$\|\eta\|_{C^{\alpha, \frac{\alpha}{r}}} \leq C_7(R)\delta^{\frac{\alpha}{r}} \|u - v\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}},$$

and as said above the short time existence of the solution to (1.3) in $C^{r+\alpha, 1+\frac{\alpha}{r}}(E_\delta)$ follows.

The uniqueness of the solution to (1.3) in $C^{r+\alpha, 1+\frac{\alpha}{r}}(E_\delta)$ follows from the above proof and a ‘standard’ argument as on p. 410 of [Lu2]. \square

Remark To guarantee the existence of the constants K_1 and K_2 in the above proof, one only needs that F and $D^\beta F$ ($|\beta| = 1$) are locally Lipschitz continuous w.r.t. $z \in Z$ and locally $C^{\alpha, \frac{\alpha}{r}}$ w.r.t. (x, t) , uniformly w.r.t. the other variables (cf. Section 8.5.3 in [Lu1]). If one uses the full power of the assumption that F is C^2 , then the above proof can be slightly simplified, cf. the proof of Theorem 3.2 in [Lu2].

4 Proof of Theorem 1.1

We need the following simple lemma which extends Lemma 8.5.5 in [Lu1].

Lemma 4.1. *Let Ω be an open set in \mathbb{R}^n with uniformly $C^{r+\alpha}$ boundary, and let $t_0 < t_1$. Let $u_i \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [t_0, t_1], \mathbb{R}^l)$ ($i = 1, 2, \dots$) be a sequence of \mathbb{R}^l -valued functions with*

$$\|u_i\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [t_0, t_1], \mathbb{R}^l)} \leq C, \quad (4.1)$$

where the constant C is independent of i . Assume that u_i converges to u in $C^0(\bar{\Omega} \times [t_0, t_1], \mathbb{R}^l)$. Then $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [t_0, t_1], \mathbb{R}^l)$ with $\|u\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [t_0, t_1], \mathbb{R}^l)} \leq C$.

Proof $\forall \Omega' \subset\subset \Omega$ with $C^{r+\alpha}$ boundary, and $\forall \beta$ with $0 < \beta < \alpha$, by the assumption (4.1) and Arzela-Ascoli theorem, there is a subsequence of $\{u_i\}$ converges to \tilde{u} in $C^{r+\beta, 1+\frac{\beta}{r}}(\bar{\Omega}' \times [t_0, t_1], \mathbb{R}^l)$, and

$$\|\tilde{u}\|_{C^{r+\beta, 1+\frac{\beta}{r}}(\bar{\Omega}' \times [t_0, t_1], \mathbb{R}^l)} \leq C.$$

By assumption u_i converges to u in $C^0(\bar{\Omega} \times [t_0, t_1], \mathbb{R}^l)$, then we get that $\tilde{u} = u$ on $\bar{\Omega}' \times [t_0, t_1]$ by the uniqueness of the limit, so $u \in C^{r+\beta, 1+\frac{\beta}{r}}(\bar{\Omega}' \times [t_0, t_1], \mathbb{R}^l)$ and

$$\|u\|_{C^{r+\beta, 1+\frac{\beta}{r}}(\bar{\Omega}' \times [t_0, t_1], \mathbb{R}^l)} \leq C.$$

By the arbitrariness of Ω' and β the desired result follows. \square

Now we prove Theorem 1.1. Let P_t and u_0 be smooth. Suppose P_0 is strongly elliptic at u_0 . By Theorem 1.2, there exists $\delta > 0$, such that the equation (1.3) has a unique solution $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(E_\delta)$ defined on $M \times [0, \delta]$. We want to show that this u is actually C^∞ -smooth. Since regularity is a local property, the desired result can be reduced to the following proposition.

Proposition 4.2. *Let $k \geq 0$ be an integer. Suppose that $\bar{\Omega}$ is a smooth, compact domain (say a ball) in \mathbb{R}^n , and Ω' is a smooth subdomain with $\bar{\Omega}' \subset \Omega$ and $\partial\bar{\Omega}'$ smooth. Let $\delta > 0$. Given $u_0 \in C^{r+k+\alpha}(\bar{\Omega})$, assume that F is a C^{2+k} -map from $\bar{\Omega} \times Z \times [0, \delta]$ to \mathbb{R}^l , where Z is a smooth, compact neighborhood of the range of $(u_0, Du_0, \dots, D^r u_0)$ in $\mathbb{R}^l \times \mathbb{R}^{nl} \times \dots \times \mathbb{R}^{n^r l}$, and there is a constant $\lambda > 0$ such that*

$$(-1)^{\frac{r}{2}-1} \frac{\partial F^a}{\partial q_{i_1 \dots i_r}^b}(x, z, t) \xi_{i_1} \dots \xi_{i_r} v^a v^b \geq \lambda |\xi|^r |v|^2 \quad (4.2)$$

uniformly w.r.t. $(x, z, t) \in \bar{\Omega} \times Z \times [0, \delta]$, where $q_{i_1 \dots i_r}^b$ are the $\mathbb{R}^{n^r l}$ -components of z , $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, and $v = (v^1, \dots, v^l) \in \mathbb{R}^l$. Let $\tilde{\delta} > 0$ and $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [0, \tilde{\delta}], \mathbb{R}^l)$ be a solution to the equation

$$\frac{\partial u}{\partial t} = F(x, u, Du, \dots, D^r u, t), \quad u(\cdot, 0) = u_0 \quad (4.3)$$

on $\bar{\Omega} \times [0, \tilde{\delta}]$. Then there exists $\delta' > 0$ such that $D^m u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{Q}')$ for all $m \leq k$, where $Q' = \Omega' \times [0, \delta']$.

Proof The proof is by a standard bootstrap procedure, compare the proof of Proposition 8.5.6 in [Lu1], Lemma 14.11 in [Li] and Theorem 8.12.1 in [K]. We do induction on k . When $k = 0$ the result is true by the hypothesis of our proposition. Suppose the proposition is true for the case $k = j$. Now consider the case $k = j + 1$. Let the hypothesis of the proposition in the case $k = j + 1$ hold. Given $u_0 \in C^{r+j+1+\alpha}(\bar{\Omega})$, let $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [0, \tilde{\delta}], \mathbb{R}^l)$ be a solution to (4.3) on $\bar{\Omega} \times [0, \tilde{\delta}]$. Choose a positive number $\delta' \leq \min\{\delta, \tilde{\delta}\}$ such that $(u(x, t), Du(x, t), \dots, D^r u(x, t)) \in Z$ for $(x, t) \in \bar{\Omega} \times [0, \delta']$. Let $x_0 \in \bar{\Omega}'$, choose $R > 0$ such that $B(x_0, 2R) \subset \Omega$. Fix i ($1 \leq i \leq n$), for $h \in \mathbb{R} \setminus \{0\}$ with $|h|$ small (not to be confused with the fiber metric h), set

$$u_h(x, t) = \frac{u(x + he_i, t) - u(x, t)}{h}, \quad (x, t) \in B(x_0, R) \times [0, \delta'],$$

where e_i is the vector in \mathbb{R}^n with the i -th component 1 and the others 0. Then u_h satisfies

$$\begin{aligned} \frac{\partial u_h^a}{\partial t} &= A_b^{aI} \partial_I u_h^b + F_h^a, \quad (x, t) \in B(x_0, R) \times [0, \delta'], \\ u_h(x, 0) &= \frac{u_0(x + he_i) - u_0(x)}{h}, \quad x \in B(x_0, R), \end{aligned}$$

where

$$\begin{aligned} A_b^{aI}(x, t) &= \int_0^1 \frac{\partial F^a}{\partial (\partial_I u^b)}(\xi_\sigma(x, t), t) d\sigma, \\ F_h^a(x, t) &= \int_0^1 \frac{\partial F^a}{\partial x^i}(\xi_\sigma(x, t), t) d\sigma, \end{aligned}$$

and

$$\xi_\sigma(x, t) = (x + \sigma he_i, \sigma(u, Du, \dots, D^r u)(x + he_i, t) + (1 - \sigma)(u, Du, \dots, D^r u)(x, t)).$$

Note that when $|h|$ is sufficiently small, $\xi_\sigma(x, t) \in \bar{\Omega} \times Z$ for $(x, t) \in B(x_0, R) \times [0, \delta']$ and $\sigma \in [0, 1]$.

Let θ be a smooth function on \mathbb{R}^n with

$$\theta \equiv 1 \text{ in } B(x_0, \frac{R}{2}), \quad \theta \equiv 0 \text{ outside } B(x_0, R).$$

Define the \mathbb{R}^l -valued function \tilde{v} on $\bar{\Omega} \times [0, \delta']$ by

$$\tilde{v}(x, t) = \theta(x)u_h(x, t) \text{ in } B(x_0, R) \times [0, \delta'], \text{ and } \tilde{v}(x, t) = 0 \text{ elsewhere.}$$

Then \tilde{v} satisfies

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} &= L_t(\tilde{v}) + \tilde{F}, \quad (x, t) \in \bar{\Omega} \times [0, \delta'], \\ \tilde{v}(x, 0) &= \tilde{v}_0(x), \quad x \in \bar{\Omega}, \\ \tilde{v}(x, t) &= 0, \quad (x, t) \in \partial\Omega \times [0, \delta'], \end{aligned}$$

where

$$\begin{aligned} (L_t \tilde{v})^a &= A_b^{aI}(x, t) \partial_I \tilde{v}^b, \\ \tilde{F} &= \theta F_h + (\theta L_t(u_h) - L_t(\theta u_h)), \end{aligned}$$

and

$$\tilde{v}_0(x) = \theta(x) \frac{u_0(x + he_i) - u_0(x)}{h} \text{ in } B(x_0, R), \text{ and } \tilde{v}_0(x) \text{ is 0 elsewhere.}$$

Observe that any $D^m u_h$ appearing in the difference $\theta L_t(u_h) - L_t(\theta u_h)$ must have $m \leq r - 1$. By assumption $u \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [0, \delta'])$, so the $C^{\alpha, \frac{\alpha}{r}}(\bar{\Omega} \times [0, \delta'])$ -norms of $D^m u_h$ ($m \leq r - 1$) are bounded by a constant independent of h . Then it follows from Theorem 4.9 of [So] that the $C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega} \times [0, \delta'])$ -norm of \tilde{v} is bounded by a constant C independent of h . (Note that \tilde{v} is 0 in a neighborhood of $\partial\Omega \times [0, \delta']$ in $\bar{\Omega} \times [0, \delta']$, so the equation for \tilde{v} satisfies the required compatibility condition there.) Clearly $u_h \rightarrow \frac{\partial u}{\partial x^i}$ in $C^0(B(x_0, R) \times [0, \delta'], \mathbb{R}^l)$ as $h \rightarrow 0$. Then by Lemma 4.1 we have $\frac{\partial u}{\partial x^i} \in C^{r+\alpha, 1+\frac{\alpha}{r}}(B(x_0, \frac{R}{2}) \times [0, \delta'])$, and $\|\frac{\partial u}{\partial x^i}\|_{C^{r+\alpha, 1+\frac{\alpha}{r}}(B(x_0, \frac{R}{2}) \times [0, \delta'])} \leq C$. It follows that there exists $R' > 0$, such that for $\Omega_1 := \{x \text{ in } \Omega \mid d(x, \bar{\Omega}') < R'\}$ we have that $v := \frac{\partial u}{\partial x^i} \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{\Omega}_1 \times [0, \delta'])$.

Now we differentiate (4.3) w.r.t. x^i and get

$$\begin{aligned} \frac{\partial v^a}{\partial t} &= \frac{\partial F^a}{\partial(\partial_I u^b)} \partial_I v^b + \frac{\partial F^a}{\partial x^i}, \quad (x, t) \in \bar{\Omega}_1 \times [0, \delta'], \\ v(x, 0) &= \frac{\partial u_0}{\partial x^i}(x), \quad x \in \bar{\Omega}_1. \end{aligned}$$

By the case $k = j$ of the proposition we have $D^m v \in C^{r+\alpha, 1+\frac{\alpha}{r}}(\bar{Q}')$ for all $m \leq j$. Then the desired result in the case $k = j + 1$ follows. \square

Note that from the above proof the number δ' that we get from the conclusion of Proposition 4.2 is independent of k in the situation of Theorem 1.1.

Acknowledgements I'm partially supported by NSFC no.11171025.

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